

# Submodule Categories of Wild Representation Type

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*Abstract.* Let  $\Lambda$  be a commutative local uniserial ring of length at least seven with radical factor ring  $k$ . We consider the category  $\mathcal{S}(\Lambda)$  of all possible embeddings of submodules of finitely generated  $\Lambda$ -modules and show that  $\mathcal{S}(\Lambda)$  is controlled  $k$ -wild with a single control object  $I \in \mathcal{S}(\Lambda)$ . In particular, it follows that each finite dimensional  $k$ -algebra can be realized as a quotient  $\text{End}(X)/\text{End}(X)_I$  of the endomorphism ring of some object  $X \in \mathcal{S}(\Lambda)$  modulo the ideal  $\text{End}(X)_I$  of all maps which factor through a finite direct sum of copies of  $I$ .

Let  $\Lambda$  be a ring. Recall that an object  $M = (M_0, M_1)$  (or written also  $(M_1 \subseteq M_0)$ ) in the submodule category  $\mathcal{S}(\Lambda)$  consists of a finitely generated  $\Lambda$ -module  $M_0$  together with a  $\Lambda$ -submodule  $M_1$  of  $M_0$ ; a morphism  $f: M \rightarrow N$  in  $\mathcal{S}(\Lambda)$  is just a  $\Lambda$ -linear map  $f: M_0 \rightarrow N_0$  which preserves the submodules, that is,  $f(M_1) \subseteq N_1$  holds.

In this paper,  $\Lambda$  always will be a commutative local uniserial ring of finite length  $n$ . Usually, we will assume that  $n \geq 7$ . The radical factor field will be denoted by  $k$  and  $t$  will be a radical generator (thus  $\Lambda/\langle t \rangle = k$ ). We have the following two special cases in mind: First of all, if  $\Lambda$  is the ring  $\mathbb{Z}/\langle p^n \rangle$  where  $p$  is a prime number, then we are dealing with the category of all possible embeddings of a subgroup in a  $p^n$ -bounded finite abelian group; the classification problem for the objects in  $\mathcal{S}(\mathbb{Z}/\langle p^n \rangle)$  was raised by Birkhoff [B] in 1934. Second, if  $\Lambda = k[T]/\langle T^n \rangle$ , where  $k[T]$  is the polynomial ring in one variable  $T$  over the field  $k$ , then we consider the possible invariant subspaces of a nilpotent operator (indeed, the objects in  $\mathcal{S}(k[T]/\langle T^n \rangle)$  may be written as triples  $(V, \phi, U)$ , where  $V$  is a  $k$ -space,  $\phi: V \rightarrow V$  is a  $k$ -linear transformation with  $\phi^n = 0$  and  $U$  is a subspace of  $V$  with  $\phi(U) \subseteq U$ ).

Some remarks concerning notions of “wildness” of additive categories will be given in the last sections. In the case  $\Lambda = \mathbb{Z}/\langle p^n \rangle$ , Arnold [A] has shown that  $\mathcal{S}(\mathbb{Z}/\langle p^{10} \rangle)$  is “wild”. In the case  $\Lambda = k[T]/\langle T^n \rangle$  Simson has shown in [Si] that  $\mathcal{S}(k[T]/\langle T^7 \rangle)$  is “wild” whereas  $\mathcal{S}(k[T]/\langle T^6 \rangle)$  is still tame, thus providing

the precise bound for “wildness”. It is not surprising that the special case  $\Lambda = k[T]/\langle T^n \rangle$  is better understood, since in this case many powerful techniques are available (in particular covering theory). The main result presented here will not dependent on  $\Lambda$  being an algebra over a field. In particular, it applies to the classical case of subgroups of finite abelian groups, as considered by Birkhoff, and it can be used in order to construct parametrized families of metabelian groups [Sc]. In case  $\Lambda$  is an algebra over a field, the last section shows in which way the main result can be strengthened.

### Controlled Wildness

Let  $\mathcal{A}$  be an additive category and  $\mathcal{C}$  a class of objects (or a full subcategory) in  $\mathcal{A}$ . Given objects  $A, A'$  in  $\mathcal{A}$ , we will write  $\text{Hom}(A, A')_{\mathcal{C}}$  for the set of maps  $A \rightarrow A'$  which factor through a (finite) direct sum of objects in  $\mathcal{C}$  (note that in this way we attach to  $\mathcal{C}$  the ideal  $\langle \mathcal{C} \rangle$  in  $\mathcal{A}$  generated by the identity morphisms of the objects in  $\mathcal{C}$ ). The same convention will apply to a single object  $C$  in  $\mathcal{A}$ : we denote by  $\text{Hom}(A, A')_C$  the set of maps  $A \rightarrow A'$  which factor through a (finite) direct sum of copies of  $C$ . Of course, given an ideal  $\mathcal{I}$  of  $\mathcal{A}$ , we write  $\mathcal{A}/\mathcal{I}$  for the corresponding factor category: it has the same objects as  $\mathcal{A}$  and given two objects  $A, A'$  of  $\mathcal{A}$ , the group  $\text{Hom}_{\mathcal{A}/\mathcal{I}}(A, A')$  is defined as  $\text{Hom}_{\mathcal{A}}(A, A')/\mathcal{I}(A, A')$ . In particular, the category  $\mathcal{A}/\langle \mathcal{C} \rangle$  has the same objects as  $\mathcal{A}$  and

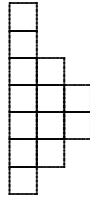
$$\text{Hom}_{\mathcal{A}/\langle \mathcal{C} \rangle}(A, A') = \text{Hom}_{\mathcal{A}}(A, A')/\text{Hom}(A, A')_{\mathcal{C}}.$$

**Definition.** We say that  $\mathcal{A}$  is *controlled  $k$ -wild* provided there are full subcategories  $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{A}$  such that  $\mathcal{B}/\langle \mathcal{C} \rangle$  is equivalent to  $\text{mod } k\langle X, Y \rangle$  where  $k\langle X, Y \rangle$  is the free  $k$ -algebra in two generators. We will call  $\mathcal{C}$  the *control class*, and in case  $\mathcal{C}$  is given by a single object  $C$  then this object  $C$  will be the *control object*. We refer to [R] for a discussion of controlled wildness.

### The Setting

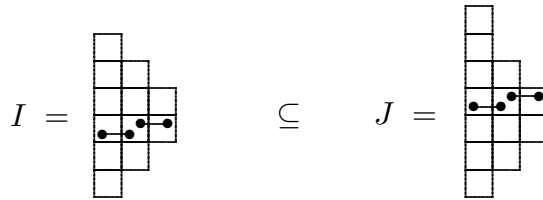
We are going to show that the category  $\mathcal{S}(\Lambda)$  is controlled  $k$ -wild. In order to do so, we need to find suitable full subcategories  $\mathcal{C} \subseteq \mathcal{B} \subseteq \mathcal{S}(\Lambda)$ . In fact,  $\mathcal{C}$  will consists of a single object  $I$ , whereas  $\mathcal{B}$  will be a suitable subcategory of the “interval” in-between the object  $I$  and a related one  $J$  with  $I \subset J$ . Given two objects  $I \subset J$  in  $\mathcal{S}(\Lambda)$ , we denote by the *interval*  $[I, J]$  the class of all objects  $M$  of  $\mathcal{S}(\Lambda)$  such that  $I^m \subseteq M \subseteq J^m$  for some natural number  $m$ .

In order to exhibit objects in  $\mathcal{S}(\Lambda)$ , it is convenient to use some graphical description. It is well-known and easy to see that the indecomposable  $\Lambda$ -modules are up to isomorphism of the form  $\Lambda/\langle t^i \rangle$  with  $1 \leq i \leq n$ , thus the indecomposable  $\Lambda$ -modules are characterized by the length ( $\Lambda/\langle t^i \rangle$  has length  $i$ ). The Krull-Remak-Schmidt theorem asserts that the isomorphism classes of the  $\Lambda$ -modules (of finite length) correspond bijectively to the partitions  $\lambda = (\lambda_1, \dots, \lambda_m)$  with all parts  $\lambda_i \leq n$ ; the  $\Lambda$ -module corresponding to the partition  $(\lambda_1, \dots, \lambda_m)$  is just  $\bigoplus_i \Lambda/\langle t^{\lambda_i} \rangle$ , or, equivalently, the  $\Lambda$ -module with generators  $x_1, \dots, x_m$  and defining relations  $t^{\lambda_i} x_i = 0$ , for  $1 \leq i \leq m$ . We will attach to a partition its Young diagram using an arrangement of boxes, however we will deviate from the usual convention as follows: the various parts will be drawn vertically and not horizontally, and the parts will not necessarily be adjusted at the top or the socle. For example, we will consider below the partition  $(7, 4, 2)$ , and it will be suitable to draw the corresponding Young diagram as follows:



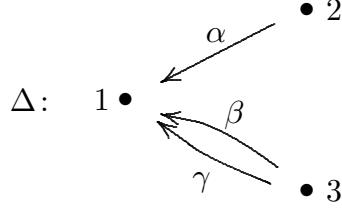
The left column has 7 boxes, the middle one 4 and the right column 2 boxes, as the partition  $(7, 4, 2)$  asserts. The adjustment of these columns made here depends on the fact that we have in mind a particular submodule, and we want that there is a generating system for the submodule such that any of these generators is a linear combination of elements which belong to boxes at the same height.

Here are the objects  $I$  and  $J$ : The  $\Lambda$ -module  $J_0$  is given by the partition  $(7, 4, 2)$ , say with generators  $x, y, z$ , annihilated by  $t^7, t^4, t^2$  respectively, and  $I_0$  is generated by  $tx, y, z$ , thus it corresponds to the partition  $(6, 4, 2)$ . The submodule  $J_1$  is generated by  $t^3x - ty$  and  $ty - z$ , and  $I_1 = tJ_1$ .



In these pictures, we have indicated the generators of the submodules using pairs of bullets which are connected by a horizontal line (and the shift of the columns was accomplished in such a way that the connecting lines become horizontal lines).

We are going to describe the interval  $[I, J]$  in-between the objects  $I$  and  $J$  in terms of representations of a quiver  $\Delta$ . The quiver  $\Delta$  looks as follows:



It has three vertices: one sink (labelled 1) and two sources (labelled 2 and 3), thus there are three simple representations  $S(1), S(2), S(3)$ . The simple representation  $S(1)$  is projective, the simple representations  $S(2)$  and  $S(3)$  are injective. Let us denote by  $\text{mod}_e k\Delta$  the full subcategory of  $\text{mod } k\Delta$  given by all representations without a simple direct summand. Note that the representations of  $\Delta$  without a simple injective direct summand are precisely the socle-projective representations (of course, a representation is said to be *socle-projective* provided the socle is projective). We denote by  $\text{mod}_{\text{sp}} k\Delta$  the full subcategory of all socle-projective representations. The inclusion functors

$$\text{mod}_e k\Delta \subset \text{mod}_{\text{sp}} k\Delta \subset \text{mod } k\Delta$$

allow us to identify the categories

$$\text{mod}_e k\Delta = \text{mod}_{\text{sp}} k\Delta / \langle S(1) \rangle = \text{mod } k\Delta / \langle S(1), S(2), S(3) \rangle,$$

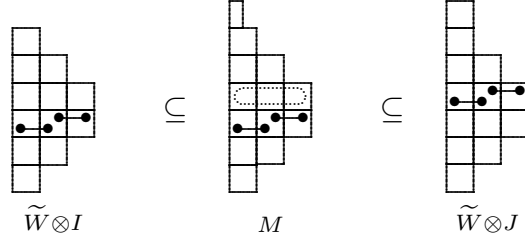
since all the simple representations of  $\Delta$  are projective or injective. Note that  $\text{mod } k\Delta / \langle S(1), S(2), S(3) \rangle$  is the factor category of  $\text{mod } k\Delta$  modulo the ideal of maps which factor through semisimple objects.

The key to proving the controlled wildness of  $\mathcal{S}(\Lambda)$  is the following result.

**Theorem 1.** *Let  $\Lambda$  be a commutative local uniserial ring of length  $n \geq 7$  and let  $k$  be its radical factor field. Then, the factor category  $[I, J] / \langle I \rangle$  is equivalent to the category  $\text{mod}_e k\Delta$ .*

The definition of  $[I, J]$  may be rephrased as follows: In order to form the direct sums  $I^m, J^m$  of copies of  $I$  and  $J$ , respectively, let  $\widetilde{W}$  be a free  $\Lambda$ -module of rank  $m$  so that we may identify the inclusion  $I^m \subset J^m$  with the map  $\widetilde{W} \otimes I \rightarrow$

$\widetilde{W} \otimes J$  induced by the inclusion  $I \subset J$ . Then each object  $M$  in  $[I, J]$  can be visualized as the middle term in a sequence of inclusions of the following type:



Here, the dotted region represents the quotient  $M_1/(\widetilde{W} \otimes_{\Lambda} I_1)$ ; and the half box on the top corresponds to the quotient  $M_0/(\widetilde{W} \otimes_{\Lambda} I_0)$ . (Note that now the boxes no longer correspond to individual composition factors of the  $\Lambda$ -module  $M_0$ , but to suitable semisimple subfactors.)

### The Layer Functors

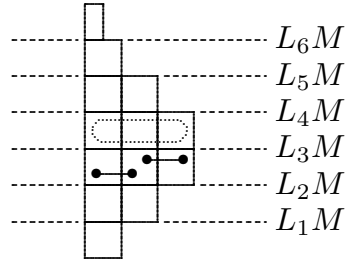
Let us analyse the objects  $M$  in  $[I, J]$ . There are the *layer functors*

$$L_i: [I, J] \rightarrow \text{mod } \Lambda$$

defined by

$$\begin{aligned} L_1 M &= t^4 M_0 \cap t^{-1} 0 \\ L_2 M &= t^3 M_0 \cap t^{-2} 0 \\ L_i M &= t^{-i+2} L_2 \quad \text{for } i \geq 3. \end{aligned}$$

Note that the  $L_i M$  are  $\Lambda$ -submodules of  $M_0$ . They form a filtration of  $M_0$  and can be visualized as follows:



This definition of the submodules  $L_i M$  only depends on  $M_0$ , it does not take into account  $M_1$ .

Of special interest is the following observation:

**Lemma.** *The subobject  $(M_1 \cap L_3M \subseteq L_6M)$  of  $M$  is a direct sum of copies of  $I$ , and any homomorphism from  $I$  to  $M$  maps into  $(M_1 \cap L_3M \subseteq L_6M)$ .*

We call this subobject  $(M_1 \cap L_3M \subseteq L_6M)$  the  $I$ -socle of  $M$ .

*Proof:* Let  $\widetilde{W}$  be a free  $\Lambda$ -module such that the inclusions  $\widetilde{W} \otimes I \subseteq M \subseteq \widetilde{W} \otimes J$  hold. The inclusion  $\widetilde{W} \otimes I \subseteq M$  embeds  $\widetilde{W} \otimes I$  into  $(M_1 \cap L_3M \subseteq L_6M)$  and clearly  $\widetilde{W} \otimes I_0 = L_6M$ . But we also have  $\widetilde{W} \otimes I_1 = \widetilde{W} \otimes tJ_1 = M_1 \cap L_3M$ . This shows that  $\widetilde{W} \otimes I = (M_1 \cap L_3M \subseteq L_6M)$ , thus  $(M_1 \cap L_3M \subseteq L_6M)$  is a direct sum of copies of  $I$ . Given a map  $I \rightarrow M$ , it will send  $I_0 = L_6I$  into  $L_6M$  and  $I_1 = I_1 \cap L_3I$  into  $M_1 \cap L_3M$ , thus it maps into  $(M_1 \cap L_3M \subseteq L_6M)$ .  $\checkmark$

### From $\mathcal{S}(\Lambda)$ to Representations of $\Delta$

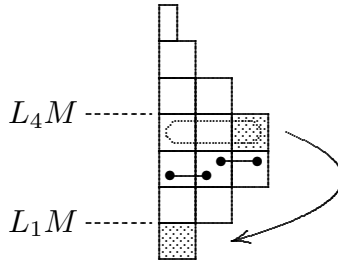
Given  $M$  in  $[I, J]$ , let  $F(M)$  be defined by

$$\begin{array}{ccc} & & M/L_6M \\ & \swarrow \alpha & \\ L_1M & & \\ & \nwarrow \beta & \\ & & M_1/(M_1 \cap L_3M) \end{array}$$

$\gamma$

with  $\alpha = t^6$ ,  $\beta = t^3$ , and a  $k$ -linear map  $\gamma$  which still has to be specified.

Actually, let us define a surjective homomorphism  $\gamma': L_4M \rightarrow L_1M$  with kernel  $L_3M + pL_5M$ , the required map  $\gamma$  will be induced by the restriction of  $\gamma'$  to  $M_1$  (note that  $M_1 \subseteq L_4M$ ). The map  $\gamma'$  will yield an isomorphism between the following two shaded boxes:



Here is the definition of  $\gamma'(c)$  for  $c \in L_4M$  in a condensed form:

$$\gamma'(c) = t^2 \left( \left( (tc + t^2 L_5M) \cap t^{-1}0 \right) + (M_1 \cap L_3M) \right) \cap t^3 L_6M$$

(note that  $\gamma'$  depends only on  $L_6$  and  $M_1 \cap L_3M$ , thus on the  $I$ -socle of  $M$ ).

In order to understand the definition and to see that  $\gamma'$  is really a  $\Lambda$ -homomorphism, we proceed stepwise: Thus, we start with  $c \in L_4M$ . Take an element  $c' \in (tc + t^2L_5M) \cap t^{-1}0$ , such an element exists since  $tL_4M = t^2L_5M + t^{-1}0$ . Next, take an element  $c'' \in (c' + (M_1 \cap L_3M)) \cap t^3L_6M$  — again, we note that such an element exists, now we use that  $c'$  belongs to  $L_3M$  and that  $L_3M = (M_1 \cap L_3M) + t^3L_6M$ . The proposed definition of  $\gamma'(c)$  amounts to  $\gamma'(c) = t^2c''$ . Now  $c'$  is unique up to the addition of elements from  $t^2L_5M \cap t^{-1}0 = t^{-1}0 \cap L_2M$ , thus  $c''$  is unique up to the addition of elements from  $(M_1 \cap L_3M) \cap t^3L_6M = L_2M$ , and the latter elements go to zero under the multiplication by  $t^2$ . This shows that  $\gamma'(c)$  is a well-defined element and since  $c''$  belongs to  $L_3M$ , we see that  $\gamma'(c)$  belongs to  $L_1M$ . Of course, it is clear that such a construction yields a homomorphism  $\gamma'$ . One finally verifies that  $\gamma'$  is surjective and that its kernel is  $L_3M + tL_5M$ .

It also follows from the construction that a homomorphism  $g: M \rightarrow N$  in  $S(\Lambda)$  between objects  $M, N \in [I, J]$  commutes with  $\gamma'$ . Hence we obtain a functor  $F: [I, J] \rightarrow \text{mod } k\Delta$ .

*Example.* Under this functor  $F$ , the object  $I$  is sent to  $F(I) = S(1)$ , whereas  $F(J)$  is the injective envelope of  $S(1)$ :

$$\begin{array}{ccc}
 & & 0 \\
 & \swarrow & \\
 F(I) = & k & \\
 & \nwarrow & \\
 & & 0
 \end{array}
 \qquad
 \begin{array}{ccc}
 & & k \\
 & \swarrow & \\
 F(J) = & k & \\
 & \nwarrow & \\
 & & k \oplus k
 \end{array}$$

**Proposition 1.** *The functor  $F$  is a full and dense functor from  $[I, J]$  onto the category  $\text{mod}_{\text{sp}} k\Delta$  of socle-projective representations of  $\Delta$ . The representation  $F(I) = S(1)$  is simple projective, and the kernel of the induced functor  $[I, J] \rightarrow \text{mod}_{\text{sp}} k\Delta / \langle S(1) \rangle$  is just the ideal of all maps which factor through a direct sum of copies of  $I$ .*

The proof of Proposition 1 will be given at the end of the next section; Theorem 1 is an immediate consequence of Proposition 1.

### ... and Back to $S(\Lambda)$

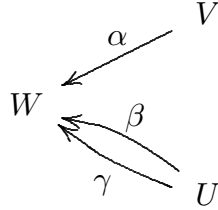
In order to show that the functor  $F$  is full and dense, we are going to present an inverse construction which we label  $\Phi$ . We work in the homomorphism category

$\mathcal{H}(\Lambda)$  for  $\Lambda$ . The objects in  $\mathcal{H}(\Lambda)$  are the  $\Lambda$ -linear maps, say  $A = (A_1 \xrightarrow{a} A_0)$ , and a morphism between two such objects  $A = (A_1 \xrightarrow{a} A_0)$  and  $B = (B_1 \xrightarrow{b} B_0)$  consists of two homomorphisms  $f_0: A_0 \rightarrow B_0$  and  $f_1: A_1 \rightarrow B_1$  such that  $f_0 a = b f_1$  holds. Clearly,  $\mathcal{S}(\Lambda)$  is just the full exact subcategory of  $\mathcal{H}(\Lambda)$  of those objects  $A = (A_1 \xrightarrow{a} A_0)$  for which the map  $a$  is monic.

Note that the inclusion  $I \rightarrow J$  gives rise to the short exact sequence in  $\mathcal{H}(\Lambda)$

$$\varepsilon: \quad 0 \longrightarrow I \longrightarrow J \longrightarrow (k \oplus k \xrightarrow{0} k) \longrightarrow 0.$$

Let  $W$  be a vector space,  $V$  a subspace of  $W$ ,  $U$  a subspace of  $W \oplus W$ , then we may consider the triple  $(W, V, U)$  as a representation of the quiver  $\Delta$  as follows



with  $\alpha$  the inclusion map, and  $\beta$  the first,  $\gamma$  the second projection of  $U$  into  $W$  (thus  $\beta(w_1, w_2) = w_1, \gamma(w_1, w_2) = w_2$ , where  $w_1, w_2 \in W$  and  $(w_1, w_2) \in U$ ). Note that in this way, we obtain precisely all the representations of  $\Delta$  which do not have a simple injective direct summand.

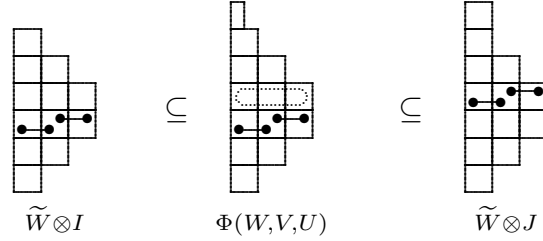
Let  $\widetilde{W}$  be a free  $\Lambda$ -module with  $\widetilde{W}/\text{rad } \widetilde{W} = W$ . In the category  $\mathcal{H}(\Lambda)$ , we consider the following fibre product construction of  $\widetilde{W} \otimes_{\Lambda} \varepsilon$  along the inclusion  $(U \xrightarrow{0} V) \longrightarrow (W \oplus W \xrightarrow{0} W)$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & \widetilde{W} \otimes_{\Lambda} I & \longrightarrow & \Phi(W, V, U) & \longrightarrow & (U \xrightarrow{0} V) \longrightarrow 0 \\ & & \parallel & & \downarrow & & \downarrow \\ 0 & \longrightarrow & \widetilde{W} \otimes_{\Lambda} I & \longrightarrow & \widetilde{W} \otimes_{\Lambda} J & \longrightarrow & (W \oplus W \xrightarrow{0} W) \longrightarrow 0 \end{array}$$

In this way, we define the object  $\Phi(W, V, U)$ . Note that by the five lemma, the vertical map in the center of the above diagram is monic, so  $\Phi(W, V, U)$ , being a subobject of an object in  $\mathcal{S}(\Lambda)$ , also lies in  $\mathcal{S}(\Lambda)$  and clearly  $F\Phi(W, V, U)$  is the subobject  $(W, V, U)$  of  $F(\widetilde{W} \otimes J) = (W, W, W \oplus W)$ .



Let us look again at the visualization of the objects in  $[I, J]$  considered above:



For such an object  $\Phi(W, V, U)$ , the dotted region represents  $U$  which is the quotient of  $\Phi(W, V, U)_1$  modulo  $\widetilde{W} \otimes_\Lambda I_1$ ; and the half box on the top corresponds to the subspace  $V$  of  $W$ , which is the quotient of  $\Phi(W, V, U)_0$  modulo  $\widetilde{W} \otimes_\Lambda I_0$ .

Suppose  $(W, V, U)$  and  $(W', V', U')$  are two such triples (thus representations of the quiver  $\Delta$  without simple injective direct summands). Let  $\widetilde{W}$  and  $\widetilde{W}'$  be free  $\Lambda$ -modules with  $\widetilde{W}/\text{rad } \widetilde{W} = W$  and  $\widetilde{W}'/\text{rad } \widetilde{W}' = W'$ , respectively. A morphism  $(W, V, U) \rightarrow (W', V', U')$  in the category  $\text{mod } \Delta$  is given by a map  $g: W \rightarrow W'$  such that  $g(V) \subseteq V'$  and  $(g \oplus g)(U) \subseteq U'$ .

Such a map  $g$  gives rise to a morphism  $\Phi(g): \Phi(W, V, U) \rightarrow \Phi(W', V', U')$  in the category  $\mathcal{S}(\Lambda)$  which makes the following diagram commutative:

$$\begin{array}{ccccccc}
0 & \longrightarrow & \widetilde{W} \otimes I & \longrightarrow & \Phi(W, V, U) & \longrightarrow & (U \rightarrow V) \longrightarrow 0 \\
& & \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
0 & \longrightarrow & \widetilde{W}' \otimes I & \longrightarrow & \Phi(W', V', U') & \longrightarrow & (U' \rightarrow V') \longrightarrow 0 \\
& & \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
0 & \longrightarrow & \widetilde{W} \otimes I & \longrightarrow & \widetilde{W} \otimes J & \longrightarrow & (W \oplus W \rightarrow W) \longrightarrow 0 \\
& & \downarrow & \searrow & \downarrow & \searrow & \downarrow \\
0 & \longrightarrow & \widetilde{W}' \otimes I & \longrightarrow & \widetilde{W}' \otimes J & \longrightarrow & (W' \oplus W' \rightarrow W') \longrightarrow 0
\end{array}$$

Indeed, one uses the projectivity of  $\widetilde{W}$  as a  $\Lambda$ -module in order to obtain a lifting  $\widetilde{g}$  of  $g$  which makes the diagram

$$\begin{array}{ccc}
\widetilde{W} & \xrightarrow{\widetilde{g}} & \widetilde{W}' \\
\text{can} \downarrow & & \downarrow \text{can} \\
W & \xrightarrow{g} & W'
\end{array}$$

commutative, then the bottom part and the right hand square in the three dimensional diagram commute. Now a fibre product construction in the category

of morphisms in  $\mathcal{H}(\Lambda)$  gives the forward pointing map in the middle of the top part of the diagram. This is the map  $\Phi(g)$  we are looking for. Since the vertical maps in the middle are both monic,  $\Phi(g): \Phi(W, V, U) \rightarrow \Phi(W', V', U')$  is just the restriction of the map  $\widetilde{W} \otimes_{\Lambda} J \xrightarrow{\widetilde{g} \otimes 1} \widetilde{W}' \otimes_{\Lambda} J$  and clearly  $F\Phi(g) = g$ .

Let us stress that the construction  $\Phi$  is not functorial, since it depends on a choice of liftings: We had to write the vector space  $W$  as  $W = \widetilde{W}/\text{rad } \widetilde{W}$  for some free  $\Lambda$ -module  $\widetilde{W}$  and given the linear transformation  $g: W \rightarrow W'$ , we used a lifting  $\widetilde{g}: \widetilde{W} \rightarrow \widetilde{W}'$  of  $g$ .

*Proof of Proposition 1:* Note that  $F(I) = (k, 0, 0)$  is the simple projective representation of  $\Delta$ . Therefore,  $F$  induces a functor  $[I, J]/\langle I \rangle$  into  $\text{mod}_{\text{sp}} k\Delta/\langle S(1) \rangle$  and this functor  $[I, J]/\langle I \rangle \rightarrow \text{mod}_{\text{sp}} k\Delta/\langle S(1) \rangle$  is full and dense. It remains to determine its kernel. For this, let  $M, M'$  be in  $[I, J]$  and consider a map  $f: M \rightarrow M'$  such that  $F(f)$  factors through a direct sum of copies of  $S(1)$ . It follows that  $F(f)_2 = F(f)_3 = 0$ . Now  $F(M) = (L_1M, M/L_6M, M_1/(M_1 \cap L_3M))$  and  $F(M') = (L_1M', M'/L_6M', M'_1/(M'_1 \cap L_3M'))$ . The maps  $F(f)_2: M/L_6M \rightarrow M'/L_6M'$  and  $F(f)_3: M_1/(M_1 \cap L_3M) \rightarrow M'_1/(M'_1 \cap L_3M')$  are induced by  $f$ ; since these are zero maps, we see that

$$f(M_0) \subseteq L_6M' \quad \text{and} \quad f(M_1) \subseteq M'_1 \cap L_3M',$$

thus  $f$  maps into the  $I$ -socle of  $M'$ . By the Lemma, the  $I$ -socle of  $M'$  is a direct sum of copies of  $I$ , thus  $f$  belongs to  $\text{Hom}(M, M')_I$ , as we wanted to show.  $\checkmark$

## Conclusion

Our main result is

**Theorem 2.** *Let  $\Lambda$  be a commutative local uniserial ring of length  $n \geq 7$  and let  $k$  be its radical factor. Then the category  $\mathcal{S}(\Lambda)$  is controlled  $k$ -wild.*

For the proof, we need the (well-known) fact that the category  $\text{mod } k\Delta$  is strictly  $k$ -wild: recall that an additive category  $\mathcal{A}$  is said to be *strictly  $k$ -wild* provided there exists a full embedding of the category  $\text{mod } k\langle X, Y \rangle$  into  $\mathcal{A}$ . The following embedding  $G$  of  $\text{mod } k\langle X, Y \rangle$  into the category of representations of  $\Delta$  is known to be full and exact: consider a  $k\langle X, Y \rangle$ -module  $(V; X, Y)$  (here,  $V$  is a  $k$ -space, and  $X$  and  $Y$  are linear transformations of  $V$ , they are given by the multiplication using the corresponding generators with the same names); under  $G$

we sent it to the following representation of  $\Delta$

$$\begin{array}{ccc}
 & V & \\
 \alpha \swarrow & & \\
 V \oplus V & \xleftarrow{\beta} & \\
 \gamma \swarrow & & \\
 & V \oplus V &
 \end{array}
 \quad \text{with} \quad
 \alpha = \begin{bmatrix} 1 \\ 0 \end{bmatrix}, \quad
 \beta = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, \quad
 \gamma = \begin{bmatrix} 0 & X \\ 1 & Y \end{bmatrix}.$$

Note that *no representation in the image of  $G$  has a simple direct summand*.

*Proof of Theorem 2:* Let  $\mathcal{B}$  be the full subcategory of all objects  $M$  in  $[I, J]$  such that either  $M = I$  or else  $F(M)$  lies in the image of the functor  $G: \text{mod } k\langle X, Y \rangle \rightarrow \text{mod}_{\text{sp}} k\Delta$ . The required equivalence  $\mathcal{B}/\langle I \rangle \rightarrow \text{mod } k\langle X, Y \rangle$  is given by the restriction of the functor  $F$  in Proposition 1 to  $\mathcal{B}$ .  $\checkmark$

Let us mention some more details of this equivalence

$$\mathcal{B}/\langle I \rangle \longrightarrow \text{mod } k\langle X, Y \rangle.$$

The  $k\langle X, Y \rangle$ -module  $(V; X, Y)$  corresponds to  $\Phi(V \oplus V, V \oplus 0, U_{XY})$  in  $[I, J]$ , where

$$U_{XY} = \{(v_1, v_2, Xv_2, v_1 + Yv_2) \mid v_1, v_2 \in V\} \subseteq V \oplus V \oplus V \oplus V;$$

the  $\Lambda$ -module  $\Phi(V \oplus V, V \oplus 0, U_{XY})_0$  is given by the partition  $(7^d, 6^d, 4^{2d}, 2^{2d})$  with  $d = \dim V_k$ , and its submodule  $\Phi(V \oplus V, V \oplus 0, U_{XY})_1$  is given by the partition  $(4^{2d}, 2^{2d})$ .

The above equivalence has the following consequence:

**Corollary.** *Let  $R$  be a finite-dimensional  $k$ -algebra. There exists  $M$  in  $\mathcal{S}(\Lambda)$  such that  $\text{End}(M)/\text{End}(M)_I$  is isomorphic to  $R$ .*

On the other hand, we stress the following (clearly also well-known) fact:

**Proposition 2.** *The category  $\mathcal{S}(\Lambda)$  is not strictly  $K$ -wild, for any field  $K$ .*

*Proof:* Assume  $\mathcal{S}(\Lambda)$  is strictly  $K$ -wild, for some field  $K$ . There are infinitely many isomorphism classes of finite length  $K\langle X, Y \rangle$ -modules  $M$  with endomorphism ring  $K$ , and there are pairs  $M, M'$  of such modules with  $\text{Hom}(M, M') =$

$0 = \text{Hom}(M', M)$ ; for example, just take for  $M$  and  $M'$  two non-isomorphic one-dimensional representations. Thus, a full embedding of  $\text{mod } K\langle X, Y \rangle$  into  $\mathcal{S}(\Lambda)$  yields an object  $(A \subseteq B)$  in  $\mathcal{S}(\Lambda)$  with endomorphism ring  $K \times K$ . Note that the multiplication with the radical generator  $t$  of  $\Lambda$  gives a nilpotent endomorphism of any object  $(A \subseteq B)$ . Thus, if  $\text{End}(A \subseteq B) = K \times K$ , then  $t$  has to act by zero on  $B$ . However, there are only two indecomposables  $(A \subseteq B)$  in  $\mathcal{S}(\Lambda)$  such that  $t$  acts as zero on  $B$ , namely

$$S_1 = (0 \subseteq k) \quad \text{and} \quad S_2 = (k \subseteq k).$$

As there are nonzero maps from  $S_1$  to  $S_2$ , it follows that  $K \times K$  cannot be realized as an endomorphism ring. ✓

### **$K$ -Algebras.**

If  $\Lambda$  is a  $K$ -algebra (for any field  $K$ , not necessarily isomorphic to the radical factor of  $\Lambda$ ) then the assignments  $\widetilde{W} = W \otimes_K \Lambda$  and  $\widetilde{g} = g \otimes_K \Lambda$  make  $\Phi: \text{mod } K\Delta \rightarrow \mathcal{S}(\Lambda)$  into a functor.

**Proposition 3.** *Assume that  $\Lambda$  is a  $K$ -algebra.*

1. *The functor  $\Phi$  is exact (and additive) and hence naturally equivalent to the tensor functor  $- \otimes_{K\Delta} \Phi(K\Delta)$ .*
2. *The composition  $F \circ \Phi$  is naturally equivalent to the identity functor on  $\text{mod } K\Delta$ , and hence  $\Phi$  preserves indecomposables and reflects isomorphisms.*
3. *The exact embedding  $\text{mod } K\langle X, Y \rangle \rightarrow \text{mod}_{\text{sp}} K\Delta \rightarrow \mathcal{S}(\Lambda)$ , makes the category  $\mathcal{S}(\Lambda)$   $K$ -wild in the sense of Drozd.* ✓

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### **References**

- [A] D. M. Arnold: *Abelian Groups and Representations of Finite Partially Ordered Sets*, Springer CMS Books in Mathematics (2000).
- [B] G. Birkhoff, *Subgroups of abelian groups*, Proc. Lond. Math. Soc., II. Ser. 38, 1934, 385–401.

- [R] C. M. Ringel, *Combinatorial representation theory. History and future* in: Representations of Algebras, Vol. I, Proc. Conf. ICRA IX, Beijing 2000; Beijing Normal University Press, 2002, 122–144.
- [Sc] M. Schmidmeier, *A construction of metabelian groups*, To appear.
- [Si] D. Simson, *Chain categories of modules and subprojective representations of posets over uniserial algebras*, Rocky Mountain J. Math. 32, 2002, 1627–1650.

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